Lecture 15

Brewster Angle, SPP, Homomorphism with Transmission Lines

Though simple that it looks, embedded in the TM Fresnel reflection coefficient are a few more interesting physical phenomena. We have looked at the physics of total internal reflection, which has inspired many interesting technologies such as waveguides, the most important of which is the optical fiber. In this lecture, we will look at other physical phenomena. These are the phenomena of Brewster's angle [115, 116] and that of surface plasmon resonance, or polariton [117, 118].

Even though transmission line theory and the theory of plane wave reflection and transmission look quite different, they are very similar in their underlying mathematical structures. For lack of a better name, we call this mathematical homomorphism (math analogy).[1] Later, to simplify the mathematics of waves in layered media, we will draw upon this mathematical homomorphism between multi-section transmission line theory and plane-wave theory in layered media.

15.1 Brewster's Angle

First, we will continue with understanding some interesting phenomena associated with the single-interface problem starting with the Brewster's angle.

[1]The use of this term could be to the chagrin of a math person, but it has also being used in a subject called homomorphic encryption or computing [119].

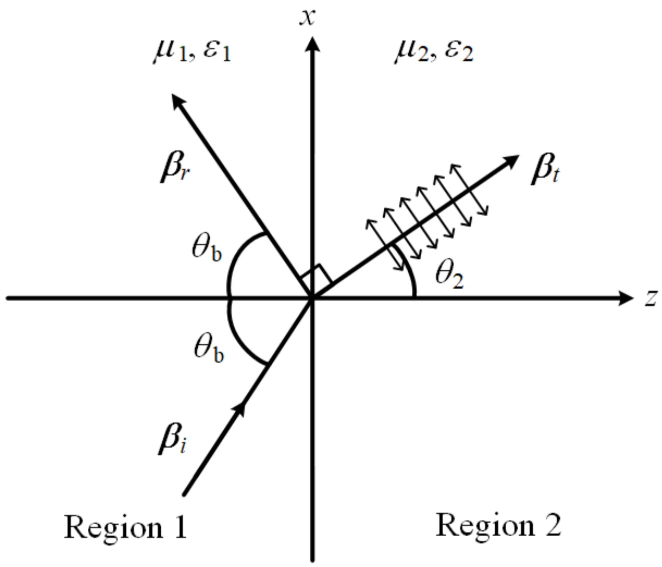

Figure 15.1: A figure showing a plane wave being reflected and transmitted at the Brewster's angle. In Region t (or Region 2), the polarization current or dipoles are all pointing in the β_r direction, and hence, there is no radiation in that direction.

Brewster angle was discovered in 1815 [115,116]. Furthermore, most materials at optical frequencies have $\varepsilon_2 \neq \varepsilon_1$, but $\mu_2 \approx \mu_1$. In other words, it is hard to obtain magnetic materials at optical frequencies. Therefore, these the TE and TM polarizations are dual to each other. Even then, the TM polarization for light behaves differently from TE polarization. Hence, we shall focus on the reflection and transmission of the TM polarization of light, and we reproduce the previously derived TM reflection coefficient here:

$$R^{TM} = \left(\frac{\beta_{1z}}{\varepsilon_1} - \frac{\beta_{2z}}{\varepsilon_2}\right) \Big/ \left(\frac{\beta_{1z}}{\varepsilon_1} + \frac{\beta_{2z}}{\varepsilon_2}\right) \tag{15.1.1}$$

The transmission coefficient is easily gotten by the formula $T^{TM} = 1 + R^{TM}$. Observe that for R^{TM}, it is possible that $R^{TM} = 0$ if

$$\varepsilon_2 \beta_{1z} = \varepsilon_1 \beta_{2z} \tag{15.1.2}$$

Squaring the above, making the note that $\beta_{iz} = \sqrt{\beta_i^2 - \beta_x^2}$, one gets

$$\varepsilon_2{}^2({\beta_1}^2 - {\beta_x}^2) = \varepsilon_1{}^2({\beta_2}^2 - {\beta_x}^2) \tag{15.1.3}$$

Solving the above, assuming $\mu_1 = \mu_2 = \mu$, gives

$$\beta_x = \omega\sqrt{\mu}\sqrt{\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1 + \varepsilon_2}} = \beta_1 \sin\theta_1 = \beta_2 \sin\theta_2 \tag{15.1.4}$$

The latter two equalities come from phase matching at the interface or Snell's law. Therefore, at the Brewster angle,

$$\sin\theta_1 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}}, \quad \sin\theta_2 = \sqrt{\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}} \tag{15.1.5}$$

or squaring the above and adding them,

$$\sin^2\theta_1 + \sin^2\theta_2 = 1, \tag{15.1.6}$$

Then, assuming that θ_1 and θ_2 are less than $\pi/2$, and using the identity that $\cos^2\theta_1+\sin^2\theta_1 = 1$, we infer that $\cos^2\theta_1 = \sin^2\theta_2$. Then it follows that

$$\sin\theta_2 = \cos\theta_1 \tag{15.1.7}$$

In other words,

$$\theta_1 + \theta_2 = \pi/2 \tag{15.1.8}$$

The above formula can be used to explain why at Brewster angle, no light is reflected back to Region 1. Figure 15.1 shows that the induced polarization dipoles in Region 2 always have their axes aligned in the direction of reflected wave. A dipole does not radiate along its axis, which can be verified heuristically by field sketch and looking at the Poynting vector. Therefore, these induced dipoles in Region 2 do not radiate in the direction of the reflected wave. Notice that when the contrast is very weak meaning that $\varepsilon_1 \cong \varepsilon_2$, then $\theta_1 \cong \theta_2 \cong \pi/4$, and (15.1.8) is satisfied.

Because of the Brewster angle effect for TM polarization when $\varepsilon_2 \neq \varepsilon_1$, $|R^{TM}|$ has to go through a null when $\theta_i = \theta_b$. Therefore, $|R^{TM}| \leq |R^{TE}|$ as shown in the plots in Figure 15.2. Then when a randomly (or arbitrarily) polarized light is incident on a surface, the polarization where the electric field is parallel to the surface (TE polarization) is reflected more than the polarization where the magnetic field is parallel to the surface (TM polarization). This phenomenon is used to design sun glasses to reduce road surface glare for drivers. For light reflected off a road surface, they are predominantly horizontally polarized with respect to the surface of the road. When sun glasses are made with vertical polarizers,[2] they will filter out and mitigate the reflected rays from the road surface to reduce road glare. This phenomenon can also be used to improve the quality of photography by using a polarizer filter as shown in Figure 15.3.

[2]Defined as one that will allow vertical polarization to pass through.

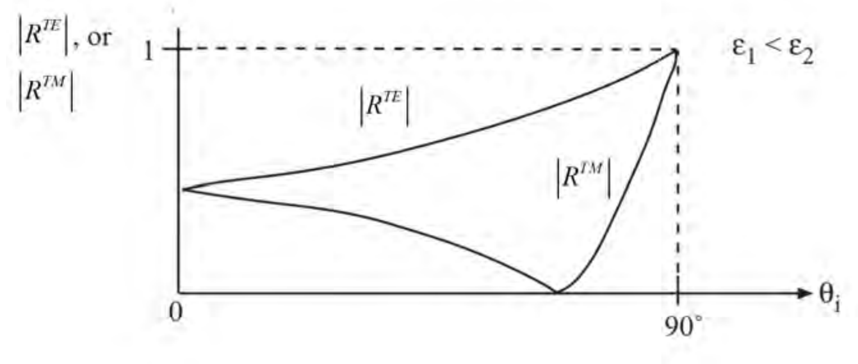

Figure 15.2: Because $|R^{TM}|$ has to through a null when $\theta_i = \theta_b$, therefore, plots of $|R^{TM}| \leq |R^{TE}|$ for all θ_i as shown above.

Figure 15.3: Because that TM and TE lights will be reflected differently, polarizer filter can produce remarkable effects on the quality of the photograph by reducing glare [116].

15.1.1 Surface Plasmon Polariton

Surface plasmon polariton occurs for the same mathematical reason for the Brewster angle effect but the physical mechanism is quite different. Many papers and textbooks will introduce this phenomenon from a different angle. But here, we will see it from the Fresnel reflection coefficient for the TM waves. When the denominator of the reflection coefficient R^{TM} is zero, it can become infinite. (This heralds the presence of some interesting physical phenomena, and in this case, a resonance behavior.) This is possible if $\varepsilon_2 < 0$, which is possible if medium 2 is a plasma medium. In this case, the criterion for the denominator to be zero is

$$-\varepsilon_2\beta_{1z} = \varepsilon_1\beta_{2z} \tag{15.1.9}$$

When R^{TM} becomes infinite, it implies that a reflected wave exists when there is no incident wave. Or when $H_{\text{ref}} = H_{\text{inc}}R^{TM}$, and $R^{TM} = \infty$, H_{inc} can be zero, and then H_{ref} can assume any value.[3] Hence, there is a plasmonic resonance or guided mode existing at the interface without the presence of an incident wave. It is a self-sustaining wave propagating in the x direction, and hence, is a guided mode propagating in the x direction.

Solving (15.1.9) after squaring it, as in the Brewster angle case, yields

$$\beta_x = \omega\sqrt{\mu}\sqrt{\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}} \tag{15.1.10}$$

This is the same equation for the Brewster angle except now that ε_2 is negative.[4] Even if $\varepsilon_2 < 0$, but $\varepsilon_1 + \varepsilon_2 < 0$ is still possible so that the expression under the square root sign (15.1.10) is positive. Thus, β_x can be pure real. The corresponding β_{1z} and β_{2z} in (15.1.9) can be pure imaginary as explained below, and (15.1.9) can still be satisfied.

This corresponds to a guided wave propagating in the x direction. When this happens,

$$\beta_{1z} = \sqrt{{\beta_1}^2 - {\beta_x}^2} = \omega\sqrt{\mu}\left[\varepsilon_1\left(1 - \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)\right]^{1/2} = \omega\sqrt{\mu}\left[\frac{\varepsilon_1^2}{\varepsilon_1+\varepsilon_2}\right]^{1/2} \tag{15.1.11}$$

Since $\varepsilon_2 < 0$, $\varepsilon_2/(\varepsilon_1+\varepsilon_2) > 1$, then β_{1z} becomes pure imaginary. Moreover, $\beta_{2z} = \sqrt{{\beta_2}^2 - {\beta_x}^2}$ and ${\beta_2}^2 < 0$ making β_{2z} becomes even a larger imaginary number. This corresponds to a trapped wave (or a bound state) at the interface. The wave decays exponentially in both directions away from the interface and they are evanescent waves.[5] This mode is shown in Figure 15.4, and is the only case in electromagnetics where a single interface can guide a surface wave, while such phenomenon abounds for elastic waves.

When one operates close to the resonance of the mode so that the denominator in (15.1.10) is almost zero, then β_x can be very large. The wavelength in the x direction becomes very short in this case, and since $\beta_{iz} = \sqrt{\beta_i^2 - \beta_x^2}$, then β_{1z} and β_{2z} become even larger imaginary numbers. Hence, the mode becomes tightly confined or bound to the interface, making the confinement of the mode very tight. This evanescent wave is much more rapidly decaying than that offered by the total internal reflection, which is $\beta_z = \sqrt{\beta_t^2 - \beta_x^2}$ where β_x is no larger than β_1. It portends use in tightly packed optical components, and has caused some excitement in the optics community.

[3]In other words, infinity times zero is undefined. This is often encountered in a resonance system like an LC tank circuit. Current flows in the tank circuit despite the absence of an exciting or driving voltage. In an ordinary differential equation or partial differential equation without a driving term (source term), such solutions are known as homogeneous solutions (to clarify the potpouri of math terms, homogeneous solutions here refer to a solution with zero source term). In a matrix equation $\overline{\mathbf{A}} \cdot \mathbf{x} = \mathbf{b}$ without a right-hand side or that $\mathbf{b} = 0$, it is known as a null-space solution.

[4]We see that it is often a dangerous proposition to square and square-root a function. We have to be guided by physical insight to see if we are finding a sane or insane solution!

[5]We have learnt about evanescent waves in Lecture 14. These waves do not carry real power.

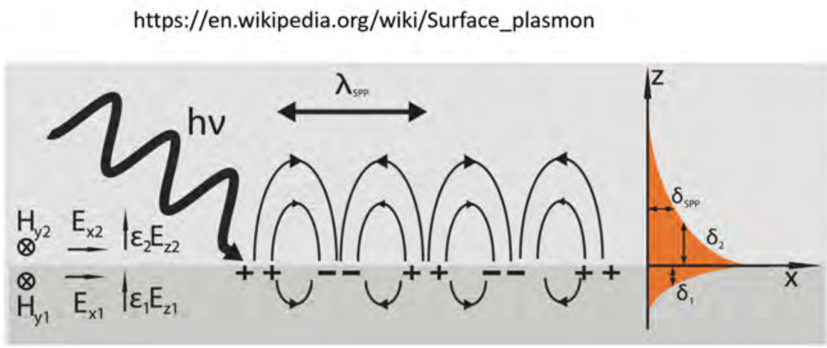

Figure 15.4: Figure showing a surface plasmonic mode propagating at an air-plasma interface. As in all resonant systems, a resonant mode entails the exchange of energies. In the case of surface plasmonic resonance, the energy is exchanged between the kinetic energy of the electrons and the energy store in the electic field (courtesy of Wikipedia [120]).

15.2 Homomorphism of Uniform Plane Waves and Transmission Lines Equations

Transmission line theory is very simple due to its one-dimensional nature. But the problem of reflection and transmission of plane waves at a planar interface is actually homormophic to that of the transmission line problem. Therefore, the plane waves through layered medium can be mapped into the multi-section transmission line problem due to mathematical homomorphism between the two problems. Hence, we can kill two birds with one stone: apply all the transmission line techniques and equations that we have learnt to solve for the solutions of waves through layered medium problems.[6]

For uniform plane waves, since they are proportional to $\exp(-j\boldsymbol{\beta}\cdot\mathbf{r})$, we know that with $\nabla \to -j\boldsymbol{\beta}$, Maxwell's equations become

$$\boldsymbol{\beta} \times \mathbf{E} = \omega\mu\mathbf{H} \tag{15.2.1}$$

$$\boldsymbol{\beta} \times \mathbf{H} = -\omega\varepsilon\mathbf{E} \tag{15.2.2}$$

for a general isotropic homogeneous medium. We will specialize these equations for different polarizations.

[6]This treatment is not found elsewhere, and is peculiar to these lecture notes.

15.2.1 TE or TE$_z$ Waves

For this, one assumes a TE wave traveling in the z direction with electric field polarized in the y direction, or $\mathbf{E} = \hat{y}E_y$, $\mathbf{H} = \hat{x}H_x + \hat{z}H_z$. Then we have from (15.2.1)

$$\beta_z E_y = -\omega\mu H_x \tag{15.2.3}$$

$$\beta_x E_y = \omega\mu H_z \tag{15.2.4}$$

From (15.2.2), we have

$$\beta_z H_x - \beta_x H_z = -\omega\varepsilon E_y \tag{15.2.5}$$

The above equations involve three variables, E_y, H_x, and H_z. But there are only two variables in the telegrapher's equations which are V and I. To this end, we will eliminate one of the variables from the above three equations. From (15.2.4), we can express H_z in terms of E_y. Then, we can show from (15.2.5) that

$$\begin{aligned}\beta_z H_x &= -\omega\varepsilon E_y + \beta_x H_z = -\omega\varepsilon E_y + \frac{\beta_x^2}{\omega\mu}E_y \\ &= -\omega\varepsilon(1 - \beta_x^2/\beta^2)E_y = -\omega\varepsilon\cos^2\theta E_y\end{aligned} \tag{15.2.6}$$

where $\beta_x = \beta\sin\theta$ has been used.

Since we still have a plane wave, Eqns. (15.2.3) and (15.2.6) can be written to look like the telegrapher's equations by letting $-j\beta_z \to d/dz$, since we still have a plane wave here. Thus,

$$\frac{d}{dz}E_y = j\omega\mu H_x \tag{15.2.7}$$

$$\frac{d}{dz}H_x = j\omega\varepsilon\cos^2\theta E_y \tag{15.2.8}$$

If we let $E_y \to V$, $H_x \to -I$, $\mu \to L$, $\varepsilon\cos^2\theta \to C$, the above is exactly analogous to the telegrapher's equations. The equivalent characteristic impedance of these equations above is then

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\cos\theta} = \sqrt{\frac{\mu}{\varepsilon}}\frac{\beta}{\beta_z} = \frac{\omega\mu}{\beta_z} \tag{15.2.9}$$

The above $\omega\mu/\beta_z$ is the wave impedance for a propagating plane wave with propagation direction or the $\boldsymbol{\beta}$ inclined with an angle θ respect to the z axis. It is analogous to the characteristic impedance Z_0 of a transmission line. When $\theta = 0$, the wave impedance $\omega\mu/\beta_z$ becomes the intrinsic impedance of space.

A two region, single-interface reflection problem can then be mathematically mapped to a single-junction connecting two-transmission-lines problem discussed in Section 13.1.1. The equivalent characteristic impedances of these two regions are then

$$Z_{01} = \frac{\omega\mu_1}{\beta_{1z}}, \quad Z_{02} = \frac{\omega\mu_2}{\beta_{2z}} \tag{15.2.10}$$

We can use the above to find Γ_{12} as given by

$$\Gamma_{12} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} = \frac{(\mu_2/\beta_{2z}) - (\mu_1/\beta_{1z})}{(\mu_2/\beta_{2z}) + (\mu_1/\beta_{1z})} \tag{15.2.11}$$

The above is the same as the Fresnel reflection coefficient found earlier for TE waves or R^{TE} after some simple re-arrangement.

Assuming that we have a single junction transmission line, one can define a transmission coefficient given by

$$T_{12} = 1 + \Gamma_{12} = \frac{2Z_{02}}{Z_{02} + Z_{01}} = \frac{2(\mu_2/\beta_{2z})}{(\mu_2/\beta_{2z}) + (\mu_1/\beta_{1z})} \tag{15.2.12}$$

The above is similar to the continuity of the voltage across the junction, which is the same as the continuity of the tangential electric field across the interface. It is also the same as the Fresnel transmission coefficient T^{TE}.

15.2.2 TM or TM$_z$ Waves

For the TM polarization, by invoking duality principle, the corresponding equations are, from (15.2.7) and (15.2.8),

$$\frac{d}{dz} H_y = -j\omega\varepsilon E_x \tag{15.2.13}$$

$$\frac{d}{dz} E_x = -j\omega\mu \cos^2\theta H_y \tag{15.2.14}$$

Just for consistency of units, since electric field is in V m^{-1}, and magnetic field is in A m^{-1} we may chose the following map to convert the above into the telegrapher's equations, viz;

$$E_y \to V, \quad H_y \to I, \quad \mu\cos^2\theta \to L, \quad \varepsilon \to C \tag{15.2.15}$$

Then, the equivalent characteristic impedance is now

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}} \cos\theta = \sqrt{\frac{\mu}{\varepsilon}} \frac{\beta_z}{\beta} = \frac{\beta_z}{\omega\varepsilon} \tag{15.2.16}$$

The above is also termed the wave impedance of a TM propagating wave making an inclined angle θ with respect to the z axis. Notice that this wave impedance again becomes the intrinsic impedance of space when $\theta = 0$.

Now, using the reflection coefficient for a single-junction transmission line, and the appropriate characteristic impedances for the two lines as given in (15.2.16), we arrive at

$$\Gamma_{12} = \frac{(\beta_{2z}/\varepsilon_2) - (\beta_{1z}/\varepsilon_1)}{(\beta_{2z}/\varepsilon_2) + (\beta_{1z}/\varepsilon_1)} \tag{15.2.17}$$

Notice that (15.2.17) has a sign difference from the definition of R^{TM} derived earlier in the last lecture. The reason is that R^{TM} is for the reflection coefficient of magnetic field while

Γ_{12} above is for the reflection coefficient of the voltage or the electric field. This difference is also seen in the definition for transmission coefficients.[7] A voltage transmission coefficient can be defined to be

$$T_{12} = 1 + \Gamma_{12} = \frac{2(\beta_{2z}/\varepsilon_2)}{(\beta_{2z}/\varepsilon_2) + (\beta_{1z}/\varepsilon_1)} \tag{15.2.18}$$

But this will be the transmission coefficient for the voltage, which is not the same as T^{TM} which is the transmission coefficient for the magnetic field or the current. Different textbooks may define different transmission coefficients for this polarization.

[7]This is often the source of confusion for these reflection and transmission coefficients.